

Thermodynamic properties of asymptotically Reissner-Nordström black holes

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Motivated by possible relation between Born-Infeld type nonlinear electrodynamics and an effective low-energy action of open string theory, asymptotically Reissner–Nordström black holes whose electric field is described by a nonlinear electrodynamics (NLED) are studied. We take into account a four dimensional topological static black hole ansatz and solve the field equations, exactly, in terms of the NLED as a matter field. The main goal of this paper is investigation of thermodynamic properties of the obtained black holes. Moreover, we calculate the heat capacity and find that the nonlinearity affects the minimum size of stable black holes. We also use Legendre-invariant metric proposed by Quevedo to obtain scalar curvature divergences. We find that the singularities of the Ricci scalar in Geometrothermodynamics (GTD) method take place at the Davies points.

I. INTRODUCTION

Recent developments on the nonlinear generalization of Maxwell electrodynamics showed that Born-Infeld (BI) type NLED may be arisen as a low energy limit of heterotic string theory. These recent progresses on the NLED theories indicated that we should consider quartic corrections of Maxwell field strength in the action of abelian electrodynamics [1–7], which led to an increased interest for BI type NLED theories.

Although NLED has been applied to various theories of black holes, one can find its interesting applications of very compact astrophysical objects. Indeed, the effects of NLED theories become quite important in super-strongly magnetized compact objects, such as pulsars, particular neutron stars, magnetars and strange quark magnetars [8–10]. Also, it has been shown that, NLED modifies in a fundamental basis the concept of gravitational redshift and its dependency of any background magnetic field as compared to the well-established method introduced by standard general relativity. Furthermore, it has been recently shown that NLED theories can remove both of the big bang and black hole singularities [11–13].

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Besides NLED applications, we should note that BI type theories are special among other classes of NLED by their remarkable properties such as birefringence phenomena, free of the shock waves [14, 15] and also enjoying an electric-magnetic duality [16]. Furthermore, considering the relation between AdS/CFT correspondence and superconductivity phenomenon, it was shown that the BI type theories make an crucial effect on the condensation, the critical temperature and energy gap of the superconductors [17].

In this paper, we investigate thermodynamic properties of the recently proposed black hole solutions of BI type models [18]. Moreover, to better understand the role of NLED, we investigate thermodynamic stability of the black hole solutions in canonical ensemble. It helps us to have a deep perspective to study how the nonlinearity affects the thermodynamic behavior of the black holes.

The plan of the paper is as follows. In Sec. II, we give a brief note of the four dimensional topological static black hole solutions. Section III is devoted to calculation of conserved and thermodynamics quantities and investigation of the first law of thermodynamics. Then, we study the thermodynamics stability in canonical ensemble and show that the nonlinearity affects the value of the horizon radius for stable black holes. This paper ends with some conclusions.

II. BLACK HOLE SOLUTIONS WITH REISSNER-NORDSTRÖM ASYMPTOTE

In Ref. [18], 4-dimensional solutions of two classes of NLED coupled to gravity was considered. This model is described by the following field equations

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \alpha \left(\frac{1}{2} g_{\mu\nu} L(\mathcal{F}) - 2 F_{\mu\lambda} F_{\nu}^{\lambda} L_{\mathcal{F}} \right), \quad (1)$$

$$\partial_{\mu} (\sqrt{-g} L_{\mathcal{F}} F^{\mu\nu}) = 0, \quad (2)$$

where $G_{\mu\nu}$ and Λ are, respectively, the Einstein tensor and the negative cosmological constant, $L_{\mathcal{F}} = \frac{dL(\mathcal{F})}{d\mathcal{F}}$. As a source of Einstein gravity, $L(\mathcal{F})$, we consider the *Exponential form of Nonlinear Electromagnetic Field* (ENEF) and the *Logarithmic form of Nonlinear Electromagnetic Field* (LNEF), in which their Lagrangians are

$$L(\mathcal{F}) = \begin{cases} \beta^2 \left(\exp(-\frac{\mathcal{F}}{\beta^2}) - 1 \right), & \text{ENEF} \\ -8\beta^2 \ln \left(1 + \frac{\mathcal{F}}{8\beta^2} \right), & \text{LNEF} \end{cases}. \quad (3)$$

The spherically symmetric solutions with a radial electric field were found in [18]. Here, we extend the solutions of Ref. [18] to the general solutions with various horizon topologies. General static

solutions may be described by the following line element

$$ds^2 = -N(r)f(r)dt^2 + \frac{dr^2}{f(r)} + r^2\check{g}_{ij}dx^i dx^j, \quad (4)$$

$$\check{g}_{ij}dx^i dx^j = \begin{cases} d\theta^2 + \sin^2\theta d\phi^2 & k = 1 \\ d\theta^2 + \sinh^2\theta d\phi^2 & k = -1 \\ d\theta^2 + d\phi^2 & k = 0 \end{cases}, \quad (5)$$

with

$$f(r) = k - \frac{2M}{r} - \frac{\Lambda r^2}{3} + \begin{cases} \frac{\beta Q}{3\sqrt{L_W}} \left[1 + L_W + \frac{4}{5}L_W^2 F\left([1], \left[\frac{9}{4}\right], \frac{L_W}{4}\right) \right], & \text{ENEF} \\ \frac{16Q^2 F\left(\left[\frac{1}{2}, \frac{1}{4}\right], \left[\frac{5}{4}\right], 1-\Gamma^2\right)}{9r^2} + \frac{4\beta^2 r^2}{9} \left[3 \ln\left(\frac{1+\Gamma}{2}\right) + 5(1-\Gamma) \right], & \text{LNEF} \end{cases}, \quad (6)$$

$$N(r) = C, \quad (7)$$

and

$$F_{tr} = -F_{rt} = \frac{Q}{r^2} \times \begin{cases} e^{-\frac{L_W}{2}}, & \text{ENEF} \\ \frac{2}{\Gamma+1}, & \text{LNEF} \end{cases}, \quad (8)$$

where $\check{g}_{ij}dx^i dx^j$ denotes the metric of two dimensional hypersurface at $r = \text{constant}$ and $t = \text{constant}$ with constant curvature $2k$ in which k is the horizon curvature constant, the integration constants Q and M are the total charge and mass of spacetime, respectively and

$$\Gamma = \sqrt{1 + \frac{Q^2}{r^4 \beta^2}},$$

$$L_W = \text{LambertW}\left(\frac{4Q^2}{\beta^2 r^4}\right).$$

Calculations of obtaining the metric functions and electromagnetic tensor have expressed in Ref. [18]. Here, we choose $N(r) = C = 1$ without loss of generality. These solutions characterize black hole configurations with various horizon properties depending on the values of the nonlinearity parameter β [18, 19]. Using the series expansion for large distance ($r \gg 1$), it has been shown that these solutions are asymptotically Reissner-Nordström [18]. Since the geometric properties of the solutions were discussed in Ref. [18], in this paper we focus on the thermodynamics properties as well as stability conditions.

III. THERMODYNAMICS PROPERTIES AND STABILITY

We compute the entropy, temperature and electric potential of the solutions in order to determine the satisfaction of the first law of thermodynamics. The entropy of Einsteinian black holes

obeys the Bekenstein-Hawking entropy area law [20–26], so the entropy associated with the event horizon is one quarter of its area, i.e.,

$$S = \pi r_+^2. \quad (9)$$

Although the (nonlinear) electromagnetic source changes the values of inner and outer horizons of charged black objects, it does not alter the areal law (see Ref. [27] for more details). Since obtained solutions are asymptotically adS, one can prove this claim by the use of the Gibbs–Duhem relation to calculate the entropy with the following approach

$$S = \frac{1}{T}(M - QU) - I, \quad (10)$$

where I is the finite total action evaluated by the use of counterterm method on the classical solutions. After cumbersome calculations, one can confirm the area law for our solutions.

The electric potential U , measured at infinity with respect to the event horizon, is [28, 29]

$$U = \begin{cases} \frac{\beta r_+ \sqrt{L_{W_+}}}{2} \left[1 + \frac{L_{W_+}}{5} F\left([1], \left[\frac{9}{4}\right], \frac{L_{W_+}}{4}\right) \right], & \text{ENEF} \\ \frac{2Q}{3r_+} \left[2F\left(\left[\frac{1}{4}, \frac{1}{2}\right], \left[\frac{5}{4}\right], 1 - \Gamma_+^2\right) - \frac{1}{(1+\Gamma_+)} \right], & \text{LNEF} \end{cases}. \quad (11)$$

In order to determine the Hawking temperature of the black hole, we use the enforcing regularity of the Euclidean section of the spacetime at the event horizon, $r = r_+$, yielding

$$T = \frac{k - \Lambda r_+^2}{4\pi r_+} + \begin{cases} \frac{\beta Q(1-L_{W_+})}{4\pi r_+ \sqrt{L_{W_+}}} - \frac{\beta^2 r_+}{8\pi}, & \text{ENEF} \\ \frac{Q^2(2-\Gamma_+)}{\pi r_+^3 \Gamma_+(\Gamma_+-1)} + \frac{\beta^2 r_+}{\pi} \left(\ln\left(\frac{\Gamma_+^2-1}{2}\right) - \frac{2}{\Gamma_+} \right), & \text{LNEF} \end{cases}, \quad (12)$$

where $\Gamma_+ = \sqrt{1 + \frac{Q^2}{r_+^4 \beta^2}}$ and $L_{W_+} = \text{LambertW}\left(\frac{4Q^2}{\beta^2 r_+^4}\right)$. In order to find the effects of nonlinearity on the temperature, we plot T for different values of β . Fig. 1 shows that there is a critical nonlinearity parameter, β_c , in which the Hawking temperature is positive definite for $\beta < \beta_c$. This situation appears for the black holes with one non-extreme horizon as it happens for Schwarzschild solutions (for more details see [18]). Moreover, we find that for $\beta > \beta_c$, there is a lower limit for the horizon radius, r_0 , in which T will be positive for $r_+ > r_0$.

Now, we are in a position to check the first law of thermodynamics. To do this, we obtain the total mass, M , as a function of the extensive quantities S and Q

$$M(S, Q) = \frac{kS^{1/2}}{2\pi^{1/2}} - \frac{\Lambda S^{3/2}}{6\pi^{3/2}} + \begin{cases} \frac{\beta Q}{6} \sqrt{\frac{S}{\pi \Gamma}} \left[1 + \Pi + \frac{4\Pi^2}{5} F\left([1], \left[\frac{9}{4}\right], \frac{\Pi}{4}\right) \right], & \text{ENEF} \\ \frac{8Q^2 F\left(\left[\frac{1}{2}, \frac{1}{4}\right], \left[\frac{5}{4}\right], 1 - \Upsilon^2\right)}{9\left(\frac{S}{\pi}\right)^{1/2}} - \frac{2\beta^2 S^{3/2} \left(\ln\left[\frac{2}{1+\Upsilon}\right] - \frac{5}{3}(1-\Upsilon) \right)}{3\pi^{3/2}}, & \text{LNEF} \end{cases}, \quad (13)$$

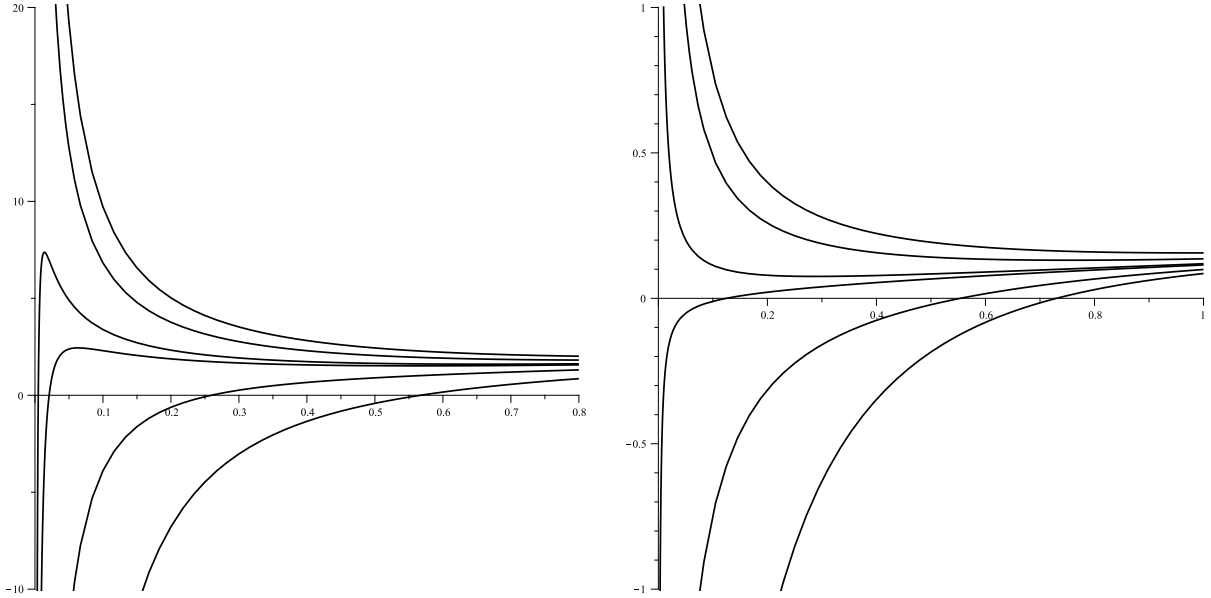


FIG. 1: Hawking temperature versus r_+ for $k = 1$, $Q = 1$, $\Lambda = -1$ and $\beta = 0.01, 0.1, 0.22, 0.26, 0.5, 1.2$ from top to down. *ENEf* (left figure) and *LNEf* (right figure)

where $\Upsilon = \sqrt{1 + \left(\frac{\pi Q}{S\beta}\right)^2}$ and $\Pi = \text{LambertW}\left(\frac{4\pi^2 Q^2}{\beta^2 S^2}\right)$.

Now, we can define the intensive parameters T and U , conjugate to extensive quantities S and Q , respectively. These intensive quantities are

$$T = \left(\frac{\partial M}{\partial S}\right)_Q, \quad U = \left(\frac{\partial M}{\partial Q}\right)_S, \quad (14)$$

After some manipulation (with numerical calculations), one can show that the temperature and electric potential calculated by Eq. (14) coincide with Eqs. (12) and (11), exactly. Eq. (14) is nothing but the first law of thermodynamics and therefore, these intensive and extensive quantities satisfy the first law

$$dM = TdS + UdQ. \quad (15)$$

A. Thermodynamic stability

Here, we investigate the thermal stability of the solutions. Depending on the set of thermodynamic variable or state functions of a system, we can examine the the thermodynamic stability from different point of views. Usually, in the canonical ensemble, in order to study the thermodynamic stability of the black holes with respect to small variations of the thermodynamic coordinates, it

is common to analyze the sign of the heat capacity at constant electric charge

$$C_Q \equiv T \left(\frac{\partial S}{\partial T} \right)_Q = T \left(\frac{\partial^2 M}{\partial S^2} \right)_Q^{-1}.$$

The positivity of the C_Q is a sufficient condition for the system to be locally stable [30, 31]. In other word, the local stability of a black hole will be ensured only if there exists a range of the event horizon radius for which the heat capacity is positive.

Using Eqs. (9), (12) and (13), and after some delicate simplifications, we obtain

$$C_Q = \begin{cases} 2\pi r_+^2 \left(\frac{\left[\Lambda r_+^2 - k + \frac{\beta^2 r_+^2}{2} \right] \sqrt{L_{W_+}} - Q\beta(1-L_{W_+})}{\left[\Lambda r_+^2 + k + \frac{\beta^2 r_+^2}{2} \right] \sqrt{L_{W_+}} - Q\beta(1+L_{W_+})} \right), & \text{ENEF} \\ 2\pi r_+^2 \left(\frac{\frac{4\beta^2 \left[\ln\left(\frac{1+\Gamma_+}{2}\right) + 1 - \Gamma_+ \right] - \Lambda + k r_+^{-2}}{4\beta^2 \left[\ln\left(\frac{1+\Gamma_+}{2}\right) - 1 + \Gamma_+ \right] - \Lambda - k r_+^{-2}}}{\right), & \text{LNEF} \end{cases}. \quad (16)$$

Numerical calculations show that there is a lower limit horizon radius for stable solutions (see Fig. 2). In other words, there is an r_{min} in which for $r_+ > r_{min}$ the heat capacity is positive and so the solutions are stable. It is easy to find that the value of r_{min} depends on the values of the metric parameters and specially the nonlinearity parameter. Fig. 2 shows that for fixed Q , k and Λ , when the value of the nonlinearity increases, the lower limit horizon radius increases, too, which means that the nonlinearity parameter affects the minimum size of stable black holes. Furthermore, in order to find the effects of NLED, one can use the series expansion of the heat capacity for large values of β , yields

$$C_Q = C_{EM} - \frac{\chi \pi Q^4 (4\Lambda r_+^4 + 2Q^2 - 3kr_+^2)}{2r_+^2 (kr_+^2 + \Lambda r_+^4 - 3Q^2)^2} \beta^{-2} + O(\beta^{-4}), \quad (17)$$

where C_{EM} is the heat capacity of Einstein Maxwell gravity with the following explicit form

$$C_{EM} = \frac{2\pi r_+^2 (\Lambda r_+^4 + Q^2 - kr_+^2)}{4(\Lambda r_+^4 - 3Q^2 + kr_+^2)},$$

and χ is equal to 8 and 1 for ENEF and LNEF branches, respectively. In Eq. (17), one finds that the second term is the leading NLED correction to the Einstein–Maxwell black hole solutions.

B. Geometrothermodynamics

In order to describe phase transitions of thermodynamic systems, one can use the concept of geometry in thermodynamics (GTD) and investigate curvature singularities so that the curvature can be interpreted as a system interaction. This method was first introduced by Weinhold [32, 33]

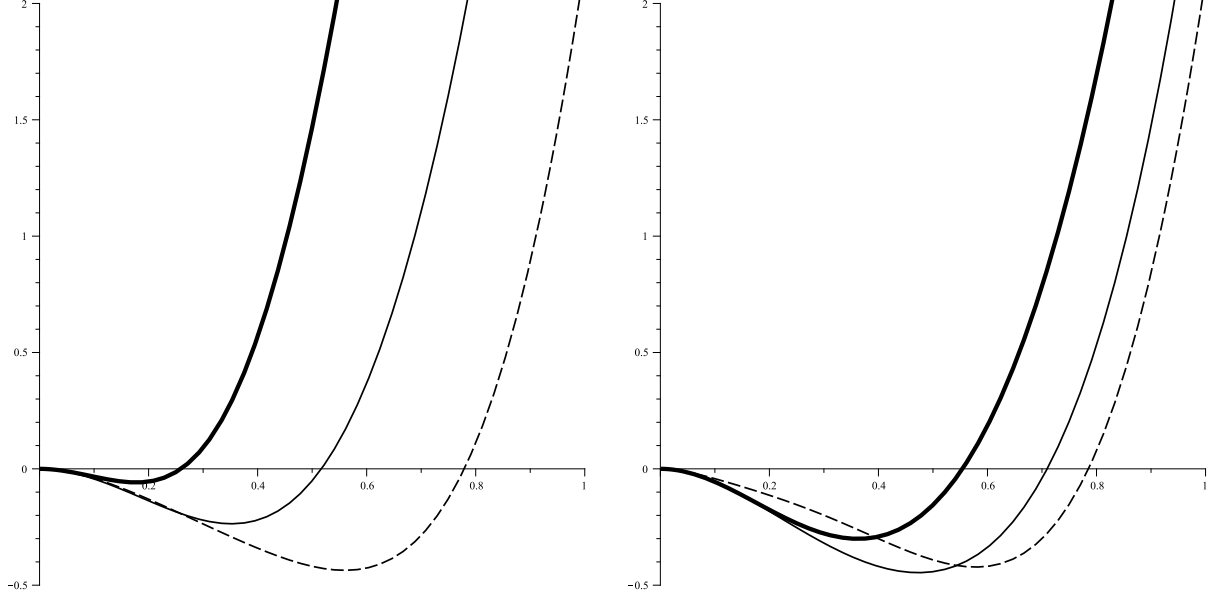


FIG. 2: Heat capacity versus r_+ for $k = 1$, $Q = 1$, $\Lambda = -1$ and $\beta = 0.5$ (bold line), $\beta = 1$ (solid line) and $\beta = 10$ (dashed line). *ENEF* (left figure) and *LNEF* (right figure)

whose Riemannian metric defined as the second derivatives of internal energy with respect to entropy and other extensive quantities (electric charge) of a thermodynamic system. After that, Ruppeiner [34, 35] introduced another metric, defined as the negative Hessian of entropy with respect to the internal energy and other extensive quantities of a thermodynamic system. We should note that the Ruppeiner metric is conformally related to the Weinhold metric with the inverse temperature as the conformal factor [36].

Sometimes, the singular points of the Weinhold and Ruppeiner metrics are not consistent with the ones of the heat capacity, unfortunately. In order to solve this puzzle, one can use the Quevedo method [37–43], whose proposed metric is Legendre-invariant. As we know, there are many Legendre-invariant metrics that one can use. Here, we can use the simplest Legendre invariant generalization of Weinhold’s metric, g^W , can be written in components as

$$g = Mg^W = M \frac{\partial^2 M}{\partial X^a \partial X^b} dX^a dX^b, \quad (18)$$

where $X^a = \{S, Q\}$ and its Legendre invariant metric can be written in terms of the Ruppeiner’s metric, g^R , as

$$g = MTg^R = -\frac{M}{\left(\frac{\partial S}{\partial M}\right)} \frac{\partial^2 S}{\partial Y^a \partial Y^b} dY^a dY^b, \quad (19)$$

where $Y^a = \{M, Q\}$. Considering Eq. (13), we use the Legendre invariant of Ruppeiner’s metric to calculate the curvature scalar. Although the method is straightforward, for reason of economy, we

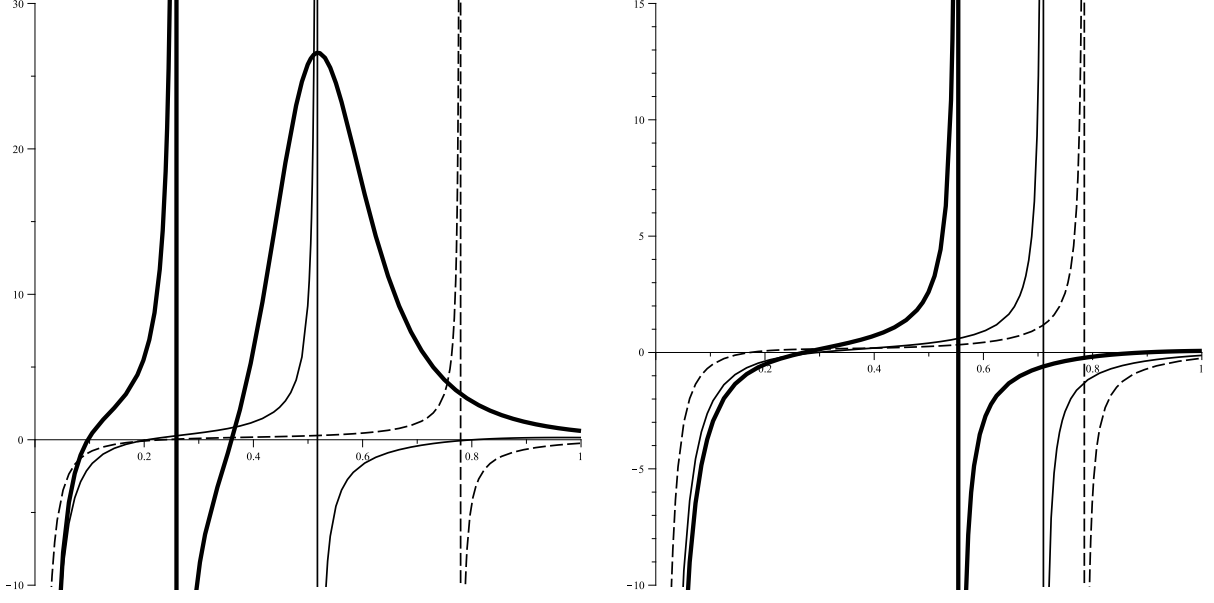


FIG. 3: Ricci scalar of GTD method versus r_+ for $k = 1$, $Q = 1$, $\Lambda = -1$ and $\beta = 0.5$ (bold line), $\beta = 1$ (solid line) and $\beta = 10$ (dashed line). *ENEf* (left figure) and *LNEf* (right figure)

do not present the analytical long equations of the Ricci scalars. We plot $R(S, Q)$ as a function of r_+ ($r_+ = \sqrt{S/\pi}$) to compare with Fig. 2. Comparing Figs. 3 and 2, we find that the singularities of the Ricci scalar (Fig. 3) take place at those points where the heat capacity vanishes (Fig. 2) and the black hole undergoes a phase transition.

IV. GENERALIZATION TO HIGHER DIMENSIONS

Here, we generalize our solutions to d -dimensional ones. Considering the following metric

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\hat{g}_k^2, \quad (20)$$

where $d\hat{g}_k^2$ is the metric of $(d-2)$ -dimensional hypersurface of constant curvature $(d-2)(d-3)k$ with the following explicit form

$$d\hat{g}_k^2 = \begin{cases} d\theta_1^2 + \sum_{i=2}^{d-2} \prod_{j=1}^{i-1} \sin^2 \theta_j d\theta_i^2 & k = 1 \\ d\theta_1^2 + \sinh^2 \theta_1 d\theta_2^2 + \sinh^2 \theta_1 \sum_{i=3}^{d-2} \prod_{j=2}^{i-1} \sin^2 \theta_j d\theta_i^2 & k = -1 \\ \sum_{i=1}^{d-2} d\phi_i^2 & k = 0 \end{cases} \quad (21)$$

Taking into account d -dimensional electromagnetic and gravitational field equations, we find

that the nonzero components of the electromagnetic field are

$$F_{tr} = -F_{rt} = \frac{Q}{r^{d-2}} \times \begin{cases} \exp\left(-\frac{L_{W_d}}{2}\right), & ENEF \\ \frac{2}{\Gamma_{d+1}}, & LNEF \end{cases}, \quad (22)$$

where

$$L_{W_d} = \text{Lambert}W\left(\frac{4Q^2}{\beta^2 r^{2d-4}}\right),$$

$$\Gamma_d = \sqrt{1 + \frac{Q^2}{\beta^2 r^{2d-4}}},$$

and d -dimensional metric function may be written as

$$f(r) = k - \frac{2\Lambda r^2}{(d-1)(d-2)} - \frac{M}{r^{d-3}} + \Sigma, \quad (23)$$

where

$$\Sigma = \begin{cases} -\frac{\beta^2 r^2}{(d-1)(d-2)} - \frac{2Q\beta}{(d-2)r^{d-3}} \int \frac{L_{W_d}-1}{\sqrt{L_{W_d}}} dr, & ENEF \\ -\frac{16\beta^2 r^2}{(d-1)(d-2)} - \frac{8\beta^2 \ln(2)}{(d-1)(d-2)} + \frac{8}{(d-2)r^{d-3}} \int \frac{\frac{Q^2}{\Gamma_{d-1}} - \beta^2 \ln\left(\frac{\beta^2 r^{2d-4}(\Gamma_{d-1})}{Q^2}\right)}{r^{d-2}} dr, & LNEF \end{cases}. \quad (24)$$

V. CONCLUSIONS

In this paper, we have considered Einstein gravity coupled to NLED in the form of exponential and logarithmic in the Lagrangian of the matter field.

Obtained solutions are generalization of topological black holes in the Einstein–Maxwell gravity and, as one expected, these solutions reduce to topological Reissner–Nordström black hole for large values of the nonlinearity parameter, β . Regarding a finite and fix value for β , one can find that the asymptotic behavior of the obtained solutions are the same as those in asymptotically adS topological Reissner–Nordström black holes.

The main goal of this paper is analyzing the thermodynamics properties of the black hole solutions. Hence, we have calculated the conserved and the thermodynamic quantities and check the validity of the first law of thermodynamics. Then, we studied the thermodynamic stability of the solutions and found that there is a lower limit for the size of stable black holes. We have plotted the heat capacity with respect to r_+ and showed that when β increases, the minimum size of stable black holes increases, too.

We also applied GTD approach proposed by Quevedo whose metric is Legendre-invariant. We used the Legendre invariant of Ruppeiner’s metric to calculate its curvature scalar and found that

the singularities of the Ricci scalar in GTD method take place at those points where the heat capacity vanishes, which is the black hole phase transition.

Last section is devoted to generalization of four dimension black holes to higher dimensional solutions. It will be straightforward to investigate thermodynamic properties of higher dimensional black holes and for reason of economy, we did not give their details.

Finally, it is worthwhile to mention that it would be interesting to generalize these Einsteinian static solutions to higher derivative gravity and rotating solution. In addition, one can investigate the non-strictly thermality of the Hawking radiation spectrum [44], quasi-normal modes and the Corda effective state of black holes [45–49] with considering the emission of Hawking quanta as a discrete process rather than a continuous process. We leave these problems for future works.

Acknowledgments

We thank Shiraz University Research Council. This work has been supported financially by Research Institute for Astronomy & Astrophysics of Maragha (RIAAM), Iran.

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